Review for Final Exam[1]

Assigned: April 26, 2021

Multivariable Calculus MATH 53
with Professor Stankova

Contents

1 Definitions

Be able to **write** precise definitions for any of the following concepts (where appropriate: both in words and in symbols), to **give** examples of each definition, and to **prove** that these definitions are satisfied in specific examples. Wherever appropriate, be able to **sketch** examples for each definition. What is/are:

1. *iterated* integral? How many such iterated integrals are there?
2. a *type I* region in the plane? A *type II*? What are they used for?
3. *switching* the *order* of integration?
4. *triple* integrals? geometric/physical interpretation?
5. *spherical* coordinates?
6. *change of variables*? the *Jacobian*?
7. *polar* change of variables? *spherical* change of variables?
8. *vector fields*? physical examples of vector fields?
9. *conservative* vector fields? testing whether a function is conservative?
10. the *line integral* of a scalar function? the geometric interpretation? *arc length*?
11. the *line integral* of a vector function? some physical interpretations?
12. the *fundamental theorem* for line integrals? *path independence* of a line integral?
13. a *connected* region in the plane? a *simply connected* region in the plane?
14. the *boundary* of a region? a *positively-oriented* boundary?
15. *Green's Theorem*?

16. the *curl* of a vector field, $\nabla \times \vec{F}$? *irrotational* vector fields?
17. the *divergence* of a vector field, $\nabla \circ \vec{F}$? *incompressible* vector fields?
18. the *Laplace* equation? a *harmonic* function?
19. a *parameterized* surface? *tangent planes* for parametric surfaces? the *normal* vector of a surface?
20. the surface area?
21. the *surface integral* of a scalar function? some physical interpretations?
22. *orientable* surfaces? an *orientation* of a surface?
23. the *surface integral* of a vector function? some physical interpretations?
24. *Stokes' Theorem*?
25. *the Divergence Theorem*?

2 Theorems

Be able to **write** what each of the following theorems (laws, propositions, corollaries, etc.) says. Be sure to understand, distinguish and **state** the conditions (hypothesis) of each theorem and its conclusion. Be prepared to **give** examples for each theorem, and most importantly, to **apply** each theorem appropriately in problems. The latter means: decide which theorem to use, check (in writing!) that all conditions of your theorem are satisfied in the problem in question, and then state (in writing!) the conclusion of the theorem using the specifics of your problem.

1. **Fubini's Theorem:** If $f(x, y)$ is continuous on a rectangle $R = [a, b]\times[c, d]$ then $\iint_R f(x, y)\,dA = \int_a^b \int_c^d f(x, y)\,dy\,dx = \int_c^d \int_a^b f(x, y)\,dx\,dy$.
$$\iint_D f(x,y)\ dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\ dy\ dx.$$
2. **Change of Variables:** If $x = f(u, v)$ and $y = g(u, v)$ have continuous partial derivatives, and these functions give a one-to-one mapping from a region S in the uv-plane to the region R in the xy-plane, then
$$\iint_R h(x, y)\,dA = \iint_S h(f(u, v), g(u, v)) \begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}\,du\,dv.$$
3. **The Three-Dimensional Jacobian:** If $x = f(u, v, w)$, $y = g(u, v, w)$, and $z = h(u, v, w)$ all have continuous partial derivatives, then the Jacobian is given by
$$\begin{vmatrix} f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{vmatrix} = f_u g_v h_w + f_v g_w h_u + f_w g_u h_v - f_u g_w h_v - f_v g_u h_w - f_w g_v h_u.$$
4. **Jacobian for Spherical Change of Variables:** If we set $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho \cos\phi$, then the Jacobian is $\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \rho^2 \sin\phi$.
5. **The Fundamental Theorem for Line Integrals:** Suppose C is a smooth curve that is parameterized by $\vec{r}(t)$ where $a \le t \le b$, and $\vec{F}$ is a continuous vector function that is conservative. Then if $\vec{F} = \nabla f$,
$$\int_C \vec{F} \circ d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$
6. **The Converse of FTL:** Suppose $\vec{F}$ is a continuous function whose domain is open and connected. Also suppose that whenever C_1 and C_2 are two smooth curves with the same start point and end point, we have $\int_{C_1} \vec{F} \circ d\vec{r} = \int_{C_2} \vec{F} \circ d\vec{r}$. Then $\vec{F}$ is conservative.
7. **A Necessary Condition for Being Conservative:** Suppose $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ has continuous partial derivatives. If there is some point in the domain where $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, then $\vec{F}$ is **not** conservative.
8. **A Sufficient Condition for Being Conservative:** Suppose $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ has continuous partial derivatives on an open, simply-connected domain. If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ at every

point in the domain, then $\vec{F}$ is conservative.

9. **Green's Theorem:** Let $\vec{F}(x,y) = \langle P, Q\rangle$ have continuous partial derivatives and no holes on the domain D. Suppose this domain is a region in the plane whose boundary is the piecewise smooth, simple curve C with positive orientation, then
$$\int_C \vec{F} \circ d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\, dA.$$
10. **A Necessary Condition for Being Conservative in 3D:** Suppose $\vec{F}(x,y,z) = \langle P, Q, R\rangle$ has continuous partial derivatives. If there is some point in the domain where curl $\vec{F} \neq 0$, then $\vec{F}$ is **not** conservative.
11. **A Sufficient Condition for Being Conservative in 3D:** Suppose $\vec{F}(x,y,z)$ has continuous partial derivatives on an open, simply-connected domain in $\mathbb{R}^3$. If curl $\vec{F} = 0$ on the domain, then $\vec{F}$ is conservative.
12. **An Exact Sequence:** For the following, suppose that all functions have continuous partial derivatives. There are three different operations that we can apply to functions in three-dimensions. Note what *type* of function (scalar or vector) each operation inputs and outputs.

$$\boxed{f(x,y,z)} \xrightarrow{\text{gradient}} \boxed{\vec{F}(x,y,z)} \xrightarrow{\text{curl}} \boxed{\vec{G}(x,y,z)} \xrightarrow{\text{divergence}} \boxed{g(x,y,z)}$$

i) **Consecutive Operations:** If we start in one of the above boxes, and perform two of the operations in a row, we will end up with the zero function.
   - The curl of the gradient of $f(x,y,z)$ equals $\langle 0,0,0\rangle$.
   - The divergence of the curl of $\vec{F}(x,y,z)$ equals 0.

ii) **Moving Backwards:** If we start with a function in one of the above boxes whose domain is *simply-connected*, and performing the operation to the right gives us the zero function, then our original function was the image of something coming from the left.
   - Suppose the domain of $\vec{F}$ is simply-connected. If the curl of $\vec{F}$ is $\langle 0,0,0\rangle$, then there is some f such that $\nabla f = \vec{F}$.
   - Suppose the domain of $\vec{G}$ is simply-connected. If the divergence of $\vec{G}$ is 0, then there is some $\vec{F}$ such that curl $\vec{F} = \vec{G}$.

iii) **The Fundamental Theorem of Calculus:** The integral of the derivative of a function over a domain equals the integral of the function over the boundary of the domain.
   - **FTL:** If $\nabla f = \vec{F}$, and C is a smooth curve in $\mathbb{R}^3$ starting at $\vec{a}$ and ending at $\vec{b}$, then
   $$\int_C \vec{F} \circ d\vec{r} = f(\vec{b}) - f(\vec{a}).$$
   - **Stokes' Theorem:** If curl $\vec{F} = \vec{G}$, and S is an oriented, piecewise smooth surface with positively-oriented boundary C, then
   $$\iint_S \vec{G} \circ d\vec{S} = \int_C \vec{F} \circ d\vec{r}.$$
   - **The Divergence Theorem:** If div $\vec{G} = g$, and E is a solid region in $\mathbb{R}^3$, whose boundary is the positively-oriented surface S, then
   $$\iiint_E g\, dV = \iint_S \vec{G} \circ d\vec{S}.$$

3 Problem Solving Techniques

1. **Switching the order of integration:** Suppose we are trying to calculate $\int_a^b \int_{g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy$, but for some reason trying to integrate with respect to x first is difficult. Then as long as $f(x,y)$ is continuous on our region, we can switch the order of integration, and find the partial antiderivative with respect to y first. However, we must be sure that we are changing the bounds of integration correctly.
   1. Draw the original region in the plane.
   2. In the above example, we started by slicing *horizontally* from $x = g_1(y)$ to $x = g_2(y)$. So switching the order of integration means we will be slicing vertically (so the integral will have $dy\,dx$).
   3. Draw some vertical lines in your region. Is there a single function $y = h_1(x)$ that describes the bottom of each of these lines? If not, we will have to break the region into multiple parts.
   4. Similarly for the top of our region. Is there a single function $y = h_2(x)$ that describes the top of our vertical lines across the region? Maybe for part of the region, the top bound looks like a line, while elsewhere it looks like a parabola, etc...
   5. Break the x-axis into different intervals $[c,d]$ based on where your vertical lines have a common lower and upper bound. And then our final integral will have a different part for each of these pieces that looks like: $\int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\,dy\,dx$.
2. **Change of Variables:** Suppose we wish to integrate $\int_R f(x,y)\,dA$, but maybe finding antiderivatives is tricky, or the region is hard to work with. We can use a multivariable version of u-substitution to simplify things, using $u = u(x,y)$ and $v = v(x,y)$.
   1. We want to solve for x as a function of u and v, and the similarly solve for y as a function of u and v. So we get two functions $x(u,v)$ and $y(u,v)$.
   2. We need to translate the region R from the xy-plane into our new variables. Like always, start by drawing the region. The boundaries of the region will give us inequalities for points in the region. For instance, we might have things like $y \geq 3x+1$ or $x \leq y^2$. Then we plug in $x(u,v)$ and $y(u,v)$ to turn these into inequalities in terms of u and v. Finally, we will solve for u or v and sketch what our new region S looks like in the uv-plane.
   3. We also need to translate dA into the new coordinates. This is done by calculating the **Jacobian**. We make a 2×2 matrix of partial derivatives $\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$, and then compute the determinant $x_u y_v - x_v y_u$. Make sure in this step that you are working with x and y as a function of u and v (and not the other way around).
   4. We can put everything together to get a double integral that we can (hopefully) evaluate directly: $\iint_S f(x(u,v), y(u,v))\,(x_u y_v - x_v y_u)\;du\,dv$.
3. **Computing the line integral of a scalar function:** Suppose we have some function $f : \mathbb{R}^2 \to \mathbb{R}$, a curve in the plane C, and we wish to calculate $\int_C f(x,y)\,ds$.
   1. Find a parameterization for the curve C, say $\vec{r}(t) = (x(t), y(t))$ and t goes from a to b.
   2. Use this to determine $ds = \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$.
   3. Then we can plug everything into our original integral to reduce it to just a single-variable integral in terms of our parameter t:
   $$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,dt.$$

   One important example of such a line integral is when $f(x,y) = 1$. Then $\int_C 1\,ds$ gives us the arc length of C.

4. **Computing the line integral of a vector function:** Suppose we have some function $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$, a curve in the plane C, and we wish to calculate $\int_C \vec{F} \circ d\vec{r}$.
   a) Check whether $\vec{F}$ is conservative. If there are no holes in the domain of $\vec{F}$, then we can check whether the y-partial derivative of the first component equals the x-partial derivative of the second component.
   b) If $\vec{F}$ is conservative, find the potential function f (so that $\nabla f = \vec{F}$). Then we can use the Fundamental Theorem to calculate the integral as $f(\text{end point}) - f(\text{start point})$.
   c) If $\vec{F}$ is *not* conservative, we have to take a "brute force" approach and reduce it down to a single-variable integral.
      i) Parameterize the curve C as some $\vec{r}(t) = \langle x(t), y(t)\rangle$ where t goes from a to b. (Make sure you keep track of the curve starting at $t = a$ and ending at $t = b$, otherwise you might get an extra negative sign).
      ii) Using the chain rule, we can calculate $d\vec{r} = \vec{r'}(t)\, dt$.
      iii) Then we can plug everything in and get a basic integral:
      $$\int_C \vec{F} \circ d\vec{r} = \int_a^b \vec{F}(x(t), y(t)) \circ \langle x'(t), y'(t)\rangle\, dt.$$
      iv) Take the dot product, and then evaluate this integral as you would in a single-variable calculus course.
5. **Using Green's Theorem to Calculate a Line Integral:** Suppose we were trying to calculate $\int_C \vec{F} \circ d\vec{r}$, but we are stuck when we try to evaluate the integral. We can use Green's Theorem to turn this into a double integral that might be easier to solve.
   1. Check that we can use Green's Theorem. We need the curve C to be enclosing some region D. We also need to make sure our function has no holes in the region D, as otherwise trying to calculate the double integral over this region doesn't work. Additionally, the curve should be *positively-oriented*. If the region enclosed by C is to the right of the curve (i.e. it is negatively-oriented), then we will multiply our final answer by -1.
   2. If $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$, calculate the mixed partial derivatives and subtract to find $Q_x - P_y$.
   3. Evaluate the double integral $\iint_D Q_x - P_y\, dA$.
6. **Using Green's Theorem to Calculate a Double Integral:** We can also use Green's Theorem in the opposite direction. When we have a double integral that is tricky to solve, we can try to turn it into a line integral that could be easier. So suppose we are trying to find $\iint_D f(x, y)\, dA$.
   1. Make sure that it makes sense to use Green's Theorem. In particular, the region D should have a boundary made up of simple closed curves. And for each of these curves, make sure you choose the positive-orientation, i.e. the region should always be to the left of the curve.
   2. We want to find an "antiderivative" for $f(x, y)$. That is, we want some vector function $\vec{F}(x, y) = \langle P, Q\rangle$ such that $f(x, y) = Q_x - P_y$. There could potentially be many different vector functions that work, and all of them will work (although some might be easier than others).
   3. Then we have our boundary curve(s) C and a vector function $\vec{F}(x, y)$, and we can calculate $\int_C \vec{F} \circ d\vec{r}$ as before: parameterize C, etc...
7. **Tangent Planes and Normal Vectors for Parametric Surfaces:** Let surface S be given by the parametrization
   $$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$$
   and let S pass throught point P with position vector $\vec{r}(u_0, v_0)$. Then the tangent vectors at

P to the two grid curves of S through P are given by

$$\begin{aligned}\vec{r}_u &= \frac{\partial}{\partial u}\vec{r}(u,v) = \left\langle \frac{\partial x}{\partial u}(u_0,v_0), \frac{\partial y}{\partial u}(u_0,v_0), \frac{\partial z}{\partial u}(u_0,v_0)\right\rangle \\ \vec{r}_v &= \frac{\partial}{\partial v}\vec{r}(u,v) = \left\langle \frac{\partial x}{\partial v}(u_0,v_0), \frac{\partial y}{\partial v}(u_0,v_0), \frac{\partial z}{\partial v}(u_0,v_0)\right\rangle\end{aligned}$$

The *tangent plane* T_P to S at P is then given by the standard parametrization of a plane passing through P and containing two non-parallel vector $\vec{r}_u$ and $\vec{r}_v$. The *normal vector* $\vec{n}$ to S at P is the cross product of the above two vectors:

$$\vec{N}_P(S) = \vec{r}_u \times \vec{r}_v.$$

8. **Computing the Surface Integral of a Scalar Function:** Suppose we have a two-dimensional surface S inside of $\mathbb{R}^3$, and we have some scalar function $f(x,y,z)$. Then we can find the "total" amount of f over our surface by computing $\iint_S f(x,y,z)\,dS$.
   1. Like always, we need to parameterize the domain S. In this case, since our domain is a two-dimensional surface, we will have *two* parameters. We will get something like $\vec{r}(s,t) = \langle x(s,t), y(s,t), z(s,t)\rangle$. Also there will be some region D that s and t are allowed to be in as part of this parameterization.
   2. Then we need to change the dS in our original integral to this new coordinate system (much like when we change variables using the Jacobian).
      a) Find the partial derivatives of your parameterization, $\vec{r}_s$ and $\vec{r}_t$.
      b) Calculate the *cross product* $\vec{r}_s \times \vec{r}_t$. Recall, this is a *normal vector* for our surface.
      c) Find the length of this normal vector.
   3. Now we can plug everything in and calculate the double integral like we would normally: $\iint_D f(x(s,t), y(s,t), z(s,t))|\vec{r}_s \times \vec{r}_t|\,dA.$
9. **Computing the Surface Integral of a Vector Function:** Suppose we have a two-dimensional surface S inside of $\mathbb{R}^3$, and we have some vector function $\vec{F}(x,y,z)$. We want to compute $\iint_S \vec{F} \circ dS$.
   1. Parameterize the surface S by $\vec{r}(s,t) = \langle x(s,t), y(s,t), z(s,t)\rangle$ and (s,t) is in D. In this situation, the *orientation* of our surface will be important.
   2. Calculate the *normal* vectors on our surface using the parameterization. (Remember there are two directions that a normal vector to a surface could point; these are the possible orientations for our surface.) To calculate the normal vector:
      a) Calculate the partial derivatives of the parameterization, $\vec{r}_s$ and $\vec{r}_t$.
      b) Take the *cross product* of these $\vec{r}_s \times \vec{r}_t$. Recall, this is a *normal vector* for our surface.
      c) Then we will have $dS = \vec{r}_s \times \vec{r}_t\,dA$.
   3. Finally we can plug everything and evaluate the basic double integral: $\iint_D \vec{F}(x(s,t), y(s,t), z(s,t)) \circ (\vec{r}_s \times \vec{r}_t)\ dA.$
10. **Calculating Curl:** If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, R exist, then

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}.$$

    Note that $\operatorname{curl}\vec{F}$ is another vector field associated to the original vector field $\vec{F}$.
11. **Finding a Potential Function for $\vec{F}$:** Suppose we have some $\vec{F}(x,y) = \langle P, Q\rangle$ and we are trying to find a potential function so that $\nabla f = \vec{F}$.

1. Check that our task is even possible. Not every vector function is conservative, so we don't want to waste our time trying to find something that doesn't exist. If the domain of $\vec{F}$ is simply connected (for instance, maybe the domain is all of $\mathbb{R}^2$), then we can just check whether or not $Q_x - P_y = 0$.
2. Once we know that $\vec{F}$ is conservative, we can start trying to find a potential function, working one component at a time. For instance, it should be the case that $\frac{\partial f}{\partial x} = P$, so we can take the x-partial antiderivative of P as our first guess for f. But we must remember the constants when we take antiderivatives, and in this case, anything with just y in it will act like a constant (with respect to x).
3. So we find the x-partial antiderivative of P as $f(x,y)+c(y)$ (where f will be explicit, but we don't know $c(y)$ yet). Then the next thing we know is that the y-partial derivative of our guess should be equal to Q. So we can calculate $\frac{\partial}{\partial y}\left(f(x,y)+c(y)\right) = Q$, and then we will be able to solve for $\frac{\partial c}{\partial y}$.
4. Then finally, we can take the y-antiderivative of $c_y(y)$ (from the previous part), and this will tell us what $c(y)$ should look like.

4 Problems for Review

The exam will be based on Homework, Lecture, Section and Quiz problems. Review **all** homework problems, and all your classnotes and discussion notes. Such a thorough review should be enough to do well on the exam. If you want to give yourself a mock-exam, select 4 representative problems from various HW assignments, give yourself 40 minutes, and then compare your solutions to the HW solutions. If you didn't manage to do some problems, analyze for yourself what went wrong, which areas, concepts and theorems you should study in more depth, and if you ran out of time, think about how to manage your time better during the upcoming exam.

4.1 Multiple Integrals

1. True/False practice:
   a) For a continuous function f, the value $\iint_R f(x,y)\,dA$ equals the area of the region R.
   b) For a continuous function f, the value $\iint_R f(x,y)\,dA$ equals the average value of f on the region R.
   c) If $f(x,y)$ is an odd function, then $\iint_R f(x,y)\,dA = 0$.
   d) Every region in the plane is either Type I or Type II.
2. For the given region R, set up the integral $\iint_R f(x,y)\,dA$ for both orders of integration.
   a) $R = \{(x,y) \mid 0 \le x \le 1, -x \le y \le x\}$.
   b) R is the sector of the unit circle between the x-axis and $30\,\text{deg}$ counterclockwise from the x-axis.
   c) R is bounded by the lines $x=2$, $x=-2$, $y=2$, $y=-2$, $xy=1$ and $xy=-1$.
3. Find the average value of $f(x,y) = x^2y$ over the triangle whose end points are $(0,0)$, $(1,2)$ and $(2,1)$.
4. Suppose D is the region in $\mathbb{R}^3$ given by: $x \ge 0$, $y \ge 0$, $z \ge 0$, $x \le y$ and $x+y+z \le 1$. Find the volume of this region using a triple integral.

4.2 Change of Variables

1. True/False practice:
   a) When evaluating $\int_{-1}^{1} f(x)\,dx$, if we use the u-substitution $u = x^2$, we can see that the integral $\int_{-1}^{1} f(x)\,dx = \int_{1}^{1} f(\sqrt{u})\frac{1}{2\sqrt{u}}\,du = 0$, since the upper and lower bound are the same.
   b) By changing to spherical coordinates, we can calculate the volume of the unit sphere as $\int_0^1 \int_0^{2\pi} \int_0^{2\pi} \rho^2 \sin\phi\,d\theta\,d\phi\,d\rho$.
2. Sketch the region bounded by $r = \cos\theta$ where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Find the integral of $f(x, y) = \frac{1}{x}$ on this region.
3. Suppose D is the disc centered at the origin with radius a. What is the average distance from the origin of a point on this disc?
4. Suppose R is the parallelogram whose corners are $(-1, -1)$, $(1, 0)$, $(0, 1)$, and $(2, 2)$. Find a change of variables that transforms the region to the unit square $[0, 1] \times [0, 1]$. Use this change of variables to evaluate $\iint_R x^2\,dA$.
5. Find $\int_0^1 \int_0^1 \frac{x-y}{x+y}\,dA$ using the change of variables $u = x - y$ and $v = x + y$.
6. The density of a block of metal is given by $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2}}$. If you cut out a solid by taking the top half of the unit sphere centered at the origin, what is the total mass of the piece you end up with?

4.3 Vector Fields

1. True/False practice:
   a) The gradient of the gradient of $f(x, y)$ is always equal to 0.
   b) If $\vec{F}(x, y)$ is a vector field, there is *at most* one function f so that $\nabla f = \vec{F}$.
   c) The gradient of $f(x, y)$ always points parallel to the level curves of $f(x, y)$.
2. For the given function f, find the domain, sketch 5 level curves and sketch ∇f.
   a) $f(x, y) = x^2 - y^2$.
   b) $f(x, y) = \ln xy$.
   c) $f(x, y, z) = xz$.
3. If $f(0, 0) = 2$ and $\nabla f(x, y) = \langle e^{xy} - y^2 + x + 2, e^{xy} - 2yx + 3\rangle$, estimate $f(0.1, 0.1)$.
4. For the following vector fields, determine whether they are *conservative*. If it is conservative, find a potential function.
   a) $\vec{F}(x, y) = \langle x^2, y^2\rangle$.
   b) $\vec{F}(x, y) = \langle y^2, x^2\rangle$.
   c) $\vec{F}(x, y, z) = \langle 8xyz^3, 4x^2z^3 + 2yz, 12x^2yz^2 + y^2 + 1\rangle$.
   d) $\vec{F}(x, y, z) = \langle 6x^2yz, 2x^3z, 3x^3y\rangle$.

4.4 Line Integrals

1. True/False practice:
   a) If we can find two different paths between the same points that give us different values for $\int_C \vec{F} \circ d\vec{r}$, then we can conclude that $\vec{F}$ is not conservative.
   b) If C is a curve between $(1, 1)$ and $(2, 3)$ then $\int_C f\,ds$ will change signs depending on which point we choose as the start or end of our parameterization.
   c) If we parameterize the curve C by moving twice as fast as someone else, then our $\int_C \vec{F} \circ d\vec{r}$ will be twice as big as theirs.

2. Find $\int_C f\,ds$.
   a) $f(x,y) = x^2 y$ and C is the line from $(0,0)$ to $(2,1)$.
   b) $f(x,y) = e^x$ and C is the bottom half of the unit circle between $(-1,0)$ and $(1,0)$.
   c) $f(x,y,z)$ is the distance from the origin and C is the helix given by $\langle \cos \pi t, \sin \pi t, t \rangle$ from $(1,0,0)$ to $(-1,0,1)$.
3. Find $\int_C \vec{F} \circ d\vec{r}$.
   a) $F(x,y) = \langle x^2, 2xy \rangle$ and C is the line from $(-1,1)$ to $(2,3)$.
   b) $F(x,y) = \langle e^x, e^y \rangle$ and C is the top half of the unit circle from $(1,0)$ to $(-1,0)$.
   c) $F(x,y,z) = \langle y, z, x \rangle$ and C is the intersection of the surfaces $z = x^2 + y^2$ and $x = y$ from the origin to $(1,1,2)$.
4. A particle travels along the level curve C given by $\sin x + e^y = 7$ from the point $(0, \ln 7)$ to $\left(\frac{pi}{2}, \ln 6\right)$. Find $\int_C \cos x\,dx + e^y\,dy$.
5. The height of a mountain is described by $h(x,y) = x^2 y + e^{xy} + x$. Suppose some walks along the path C from $(0,0)$ to $(2,2)$ determined by making sure at every point, the person is moving in the direction where the slope is the steepest. What is $\int_C \nabla h \circ d\vec{r}$?

4.5 Green's Theorem

1. True/False practice:
   a) If $\vec{F}(x,y)$ has a hole at the origin, then Green's Theorem will fail for any region in the plane.
   b) A positively-oriented curve is one that travels counterclockwise in the plane.
2. Suppose C is the triangular path moving from $(0,0)$ to $(2,1)$ to $(1,2)$ and then back to the origin. Using Green's Theorem, find $\int_C \langle e^x \sin x + y, y^2 - x^2 \rangle$.
3. Find $\int_C x^2 y\,dx - xy^2\,dy$ where C is the circle of radius 2 centered at the origin traveling clockwise.
4. Suppose R is the region between $y = \sin x$ and the x-axis from $x = 0$ to $x = \pi$. Use Green's Theorem to find $\iint_R 2y\,dA$.
5. Suppose C is the ellipse given by $25x^2 + 9y^2 = 1$ and $\vec{F}(x,y) = \langle y \ln(x^2+y^2), x \ln(x^2+y^2) \rangle$. (Hint: It'd be much easier to integrate this over the unit circle than the ellipse...)

4.6 Surface Integrals

1. Suppose S is the triangle formed by the plane $x + y + z = 1$ between the points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Using a double integral, find the area of this surface.
2. Suppose E is the solid formed by intersecting the cylinder $x^2 + y^2 \leq 1$ with the planes $z = 0$ and $z = x + y + 5$. Find the total surface area of this solid.
3. Find the surface area of the paraboloid $z = x^2 + y^2$ inside the cylinder $x^2 + y^2 = 9$.
4. Suppose S is the triangle formed by the plane $x + y + z = 1$ between the points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.
   a) Find $\iint_S xy\,dS$.
   b) Find $\iint_S \langle x, y, z \rangle \circ d\vec{S}$.
5. Suppose the surface S is the top half of the unit sphere centered at the origin.
   a) If the density is given by $f(x,y,z) = 2x + 3y + z$, find the total mass of this surface.
   b) If $\vec{F}(x,y,z) = \langle x^2, 0, y^2 \rangle$, find the flux of $\vec{F}$ across S.

4.7 Gradient, Curl, Divergence

1. True/False practice:
   a) The curl of the curl of $\vec{F}$ is always $\langle 0,0,0\rangle$.
   b) The curl of the curl of $\vec{F}$ is never $\langle 0,0,0\rangle$.
2. For the following vector fields, compute the curl and the divergence. Determine whether $\vec{F}$ is the gradient of some f, and find such a function if possible. Determine whether $\vec{F}$ is the curl of some function $\vec{G}$, and find such a function if possible.
   a) $F(x,y,z) = \langle x^2, y^2, z^2\rangle$
   b) $F(x,y,z) = \langle \sin y, \sin z, \sin x\rangle$
   c) $F(x,y,z) = \langle xyz, xyz, xyz\rangle$
   d) $F(x,y,z) = \langle 1,1,1\rangle$
3. Show that $\nabla \circ (\vec{F} \times \vec{G}) = (\nabla \times \vec{F}) \circ \vec{G} - \vec{F} \circ (\nabla \times \vec{G})$.
4. Show that *any* continuous scalar function $f(x,y,z)$ is equal to the divergence of some vector field.
5. Suppose $f(x,y)$ and $g(x,y)$ have continuous partial derivatives, C is the unit circle oriented counterclockwise, and D is the region inside the unit circle. If $\vec{n}$ is the unit normal vector for the unit circle, using Green's Theorem, it is possible to prove that

$$\int_C f \cdot (\nabla g \circ \vec{n})\, ds = \iint_D \nabla f \circ \nabla g\, dA + \iint_D f \cdot \left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}\right) dA.$$

   Suppose $h(x,y)$ is a harmonic function, and $h(x,y) = 0$ for any point on the unit circle. Use the above fact to show that $\iint_D |\nabla h|^2\, dA = 0$.

4.8 Stokes' Theorem and the Divergence Theorem

1. Use Stokes' Theorem to evaluate the line integral $\int_C \vec{F} \circ d\vec{r}$ where C is the triangle from $(1,0,0)$ to $(0,1,0)$ to $(0,0,1)$ to $(1,0,0)$ and $\vec{F}(x,y,z) = \langle x+y^2, y+z^2, z+x^2\rangle$.
2. Let $\vec{G} = \langle z, x, -x\rangle$ and S is the surface of the paraboloid $z = x^2+y^2$ from $z=0$ to $z=9$ oriented so that the normal vectors point outside.
   a) What is the positively-oriented boundary of S?
   b) Show that $\vec{G}$ is the curl of something.
   c) Use Stokes' Theorem to calculate $\iint_S \vec{G} \circ d\vec{S}$.
3. Let $\vec{F} = \langle y+z, 2x, 2\rangle$ and E is the solid given by the cone $y^2 = x^2+z^2$ between $y=0$ and $y=4$.
   a) What is the positively-oriented boundary of E?
   b) If S is this boundary, calculate $\iint_S \vec{F} \circ d\vec{S}$ directly.
   c) Use the Divergence Theorem to calculate $\iint_S \vec{F} \circ d\vec{S}$.

5 No Calculators during the Exam. Cheat Sheet and Studying for the Exam

No calculators are allowed on the exam. Anyone caught using a calculator will be disqualified from the exam.

For the exam, you are allowed to have a "cheat sheet" - *two pages* of a regular 8.5×11 sheet. You can write whatever you wish there, under the following conditions:

- The whole cheat sheet must be **handwritten by your own hand**! No xeroxing, no copying, (and for that matter, no tearing pages from the textbook and pasting them onto your cheat sheet.) DSP students with special writing or related disability should consult with the instructor regarding their cheat sheets.
- You must submit your cheat sheet on Gradescope an hour **before** the start time of your exam.

- Any violation of these rules will disqualify your cheat sheet and may end in your own disqualification from the exam. I may decide to randomly check your cheat sheets, so let's play it fair and square. :)
- Don't be a **freakasaurus**! Start studying for the exam several days in advance, and prepare your cheat sheet at least 2 days in advance. This will give you enough time to become familiar with your cheat sheet and be able to use it more efficiently on the exam.
- **Do NOT overstudy on the day of the exam!! No sleeping the night before the exam due to cramming, or more than 3 hours of math study on the day of the exam is counterproductive! No kidding!**